Inequalities Involving Immanants

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ABSTRACT

We summarize the recent progress toward determining the ordering of the normalized immanants regarded as generalized matrix functions and restricted to the $n \times n$ positive semidefinite Hermitian matrices. Included are the most recent results of Heyfron and Pate. Moreover, we describe some of the techniques used.

1. INTRODUCTION

If c is a function from S_n to $\mathbb{C}$, where S_n denotes the symmetric group on $\{1, 2, \ldots, n\}$, then we define the matrix function $d_c(\ )$ by

$$d_c(A) = \sum_{\sigma \in \mathrm{S}_n} c(\sigma) \prod_{t=1}^{n} a_{t,\sigma(t)}$$

for each $n \times n$ complex matrix $A = [a_{ij}]$. For example, if $c(\sigma) = 1$ for each $\sigma \in \mathrm{S}_n$, then $d_c(\ )$ is the permanent function per( ), while if ϵ denotes the signum function, then $d_c(\ )$ is det( ), the determinant function.

If G is a finite group, then the set of all functions from G to $\mathbb{C}$, denoted by $\mathbb{C}G$, is an algebra known as a group algebra. Addition and scalar multiplication are defined in the expected manner on $\mathbb{C}G$, while multiplication of $f, g \in \mathbb{C}G$ is defined by

$$(fg)(\sigma) = \sum_{\tau \in G} f(\sigma\tau^{-1})g(\tau)$$

for each $\sigma \in G$. We shall endow our group algebras $\mathbb{C}G$ with an involution

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$f \to f^*$ defined by $f^*(\sigma) = \overline{f(\sigma^{-1})}$ for each $\sigma \in G$. Members f of $\mathbb{C}G$ such that $f^* = f$ are said to be Hermitian.

We shall be concerned with the restriction of the functions $d_c(\ )$, where $c \in \mathbb{C}\mathrm{S}_n$, to $\mathcal{H}_n$, the $n \times n$ positive semidefinite Hermitian matrices; hence, we tend to regard d as a function from $\mathbb{C}\mathrm{S}_n \times \mathcal{H}_n$ to $\mathbb{C}$.

If $f, g \in \mathbb{C}\mathrm{S}_n$, then we write $f \geq g$ if $d_f(A) \geq d_g(A)$ for each $A \in \mathcal{H}_n$. In particular, if $c \in \mathbb{C}\mathrm{S}_n$, then we shall write $c \geq 0$ if $d_c(A) \geq 0$ for each $A \in \mathcal{H}_n$. If $f \in \mathbb{C}\mathrm{S}_n$ and $f(e) \neq 0$, where e denotes the identity in S_n, then by $\widehat{f}$ we shall mean $[f(e)]^{-1}f$.

2. CLASSICAL RESULTS AND INTERESTING CONJECTURES

The following theorem of I. Schur has inspired much of the later work involving inequalities for the restriction of the matrix functions d_c to the positive semidefinite Hermitian matrices, particularly in the case where c is a character of a subgroup S_n.

THEOREM 1. *If G is a subgroup of S_n and λ is a character of G, then $d_\lambda(A) \geq \lambda(e)\det(A)$ for each $A \in \mathcal{H}_n$.*

In our notation the conclusion of Theorem 1 is simply $\widehat{\lambda} \geq \epsilon$. Note that the inequality $d_\lambda(A) \geq \lambda(e)\det(A)$ reduces to equality in case A is the $n \times n$ identity matrix. This is commonly the case with the inequalities that we will consider.

A member c of $\mathbb{C}\mathrm{S}_n$ is said to be positive semidefinite if there exists an $f \in \mathbb{C}\mathrm{S}_n$ such that $c = ff^*$. Alternately, a member c of $\mathbb{C}\mathrm{S}_n$ is positive semidefinite if and only if

$$\sum_{\sigma \in \mathrm{S}_n} \sum_{\tau \in \mathrm{S}_n} c(\sigma\tau^{-1}) x(\sigma)\overline{x(\tau)} \geq 0$$

for each $x \in \mathbb{C}\mathrm{S}_n$. If $f, g \in \mathbb{C}\mathrm{S}_n$ then $(fg)^* = g^* f^*$; hence, $(ff^*)^* = ff^*$ for each $f \in \mathbb{C}\mathrm{S}_n$. Therefore, all positive semidefinite members of $\mathbb{C}\mathrm{S}_n$ are Hermitian.

If λ is a character of the finite group G, then $\lambda(\sigma) = \overline{\lambda(\sigma^{-1})}$; hence all characters of subgroups G of S_n are Hermitian. Moreover, if λ is an irreducible character, then $\lambda^2 = [|G|/\lambda(e)]\lambda$; hence

$$\lambda\lambda^* = \lambda^2 = \frac{|G|}{\lambda(e)}\lambda.$$

The number $\lambda(e)$ is a positive integer known as the degree of the λ. Irreducible characters of subgroups of S_n are therefore positive semidefinite. Consequently, the following theorem of Merris and da Silva [9] is a generalization Theorem 1.

THEOREM 2. *If c is a positive semidefinite member of $\mathbb{C}S_n$, then $d_c(A) \geq c(e)\det(A)$ for each $A \in \mathcal{H}_n$.*

In our notation this theorem states that $\widehat{c} \geq \epsilon$ for each positive semidefinite $c \in \mathbb{C}S_n$. Schur's original proof of Theroem 1 was quite difficult, but, as one might expect, simpler proofs for it and Theorem 2 have been found. See [1].

The following popular conjecture, known as the permanental dominance conjecture, is the permanental analogue to Theorem 1. This conjecture was first published in E. H. Lieb's article [8] as Conjecture α.

CONJECTURE 1. *If G is a subgroup of S_n and λ is a character of G, then $d_\lambda(A) \leq \lambda(e)\,\mathrm{per}(A)$ for each $A \in \mathcal{H}_n$.*

Conjecture 1 is known to hold for certain special subgroups of S_n. For example, it holds for the symmetric groups S_n whenever $n \leq 9$. See [18]. Moreover, it is known to hold for the trivial character for certain special groups G. For example, suppose $1 \leq t < n$, and let G denote the subgroup of S_n consisting of all $\sigma \in S_n$ such that $\sigma(\{1,2,\ldots,t\}) = \{1,2,\ldots,t\}$. Then, if c denotes the trivial character of G (that is, the member of $\mathbb{C}S_n$ that assumes the value 1 at each member of G and the value zero elsewhere), then we have $d_c(A) \leq c(e)\,\mathrm{per}(A)$ for each $A \in \mathcal{H}_n$. Of course, this inequality is the well-known Lieb permanental inequality; see [8]. Moreover, the Lieb inequality implies that Conjecture 1 holds for the trivial character in case the subgroup G is isomorphic to a direct product of other symmetric groups. Such subgroups are known as Young subgroups.

As a further example we consider a result which was inspired by a conjecture of M. Marcus. Consider an $np \times np$ positive semidefinite Hermitian matrix A partitioned as an $n \times n$ block matrix $[A_{ij}]$ each of whose blocks A_{ij} is a $p \times p$ matrix. Let B denote the $n \times n$ matrix whose ijth entry is $\mathrm{per}(A_{ij})$. Then Marcus conjectured that $\mathrm{per}(A) \geq \mathrm{per}(B)$. Lieb, in the 1966 article referenced above, verifies the conjecture of Marcus in case $n = 2$ and points out that the conjecture of Marcus is a special case of Conjecture 1. In [14] Pate shows that if p is sufficiently large with respect to n and A is a real positive semidefinite symmetric matrix, then the conjecture of Marcus holds. Recently, Heyfron, in work as yet unpublished, has extended his theorem to the complex case and considerably shortened the author's proof by using techniques from the representation theory of wreath products.

If $A = [a_{ij}]$ is an $n \times n$ matrix, then the Π-matrix, Π_A, associated with A is the $n! \times n!$ matrix indexed by the members of S_n whose $\sigma\tau$th entry is $\prod_{t=1}^{n} a_{\sigma(t),\tau(t)}$. It is known that if $A \in \mathcal{H}_n$ then $\Pi_A \in \mathcal{H}_{n!}$; hence the numerical range of Π_A is a closed interval, $[v, w]$, contained in the nonnegative real numbers, such that v is that minimum eigenvalue of Π_A and w is the maximum eigenvalue of Π_A. Although the following result is well known, the author has not been able to find a reference. A proof will therefore be presented.

LEMMA 1. *If $A = [a_{ij}]$ is a member of $\mathcal{H}_n$ and x is a nonnegative real number, then x is a member of the numerical range of Π_A if and only if there exists a positive semidefinite $c \in \mathbb{C}S_n$ such that $c(e) = 1$ and*

$$x = \sum_{\sigma \in S_n} c(\sigma) \prod_{t=1}^{n} a_{t,\sigma(t)}.$$

Proof. Clearly, x is a member of the numerical range of Π_A if and only if there exists an $f \in \mathbb{C}S_n$ such that $\|f\| = 1$ and

$$x = \sum_{\sigma \in S_n} \sum_{\tau \in S_n} \overline{f(\sigma)} f(\tau) \prod_{t=1}^{n} a_{\sigma(t),\tau(t)}.$$

Note, however, that

$$\begin{aligned}
\sum_{\sigma \in S_n} \sum_{\tau \in S_n} \overline{f(\sigma)} f(\tau) \prod_{t=1}^{n} a_{\sigma(t),\tau(t)} &= \sum_{\sigma,\tau} \overline{f(\sigma)} f(\tau) \prod_{s=1}^{n} a_{s,\tau\sigma^{-1}(s)} \\
&= \sum_{\sigma,\mu} f(\mu\sigma) \overline{f(\sigma)} \prod_{t=1}^{n} a_{t,\mu(t)} \\
&= \sum_{\mu} \left(\sum_{\sigma \in S_n} f(\mu\sigma) f^*(\sigma^{-1}) \right) \prod_{t=1}^{n} a_{t,\mu(t)} \\
&= \sum_{\sigma \in S_n} (ff^*)(\sigma) \prod_{t=1}^{n} a_{t,\sigma(t)}.
\end{aligned}$$

Hence, if we let $c = ff^*$ then $c(e) = \|f\|^2 = 1$ and we have

$$x = \sum_{\sigma \in S_n} c(\sigma) \prod_{t=1}^{n} a_{t,\sigma(t)}$$

as required. ∎

Lemma 1 and Theorem 2 clearly imply that if $A \in \mathcal{H}_n$ then $\det(A)$ is the infimum of the numerical range of Π_A and hence the minimum eigenvalue of Π_A. It is the conjecture of G. W. Soules [13] that if $A \in \mathcal{H}_n$, then $\mathrm{per}(A)$ is the supremum of the numerical range of Π_A and hence the maximum eigenvalue of Π_A. By Lemma 1 the Soules conjecture is equivalent to the following, which is clearly the permanental analogue of Theorem 2.

CONJECTURE 2. *If $A = [a_{ij}]$ is a member of $\mathcal{H}_n$ an c is a positive semidefinite member of $\mathbb{C}S_n$, then*

$$\sum_{\sigma \in S_n} c(\sigma) \prod_{t=1}^{n} a_{t,\sigma(t)} \le c(e)\ \mathrm{per}(A).$$

If λ is an irreducible character of S_n, then the matrix function d_λ is called an immanant. Though efforts to resolve Conjecture 1 in its most general form have produced very little, considerable progress has been on the permanental dominance conjecture in the special case $G = S_n$. This conjecture, which we call the permanental dominance conjecture for immanants and which may be true even if Conjecture 1 is false, is recorded below.

CONJECTURE 3. *If λ is an irreducible character of S_n, then $d_\lambda(A) \le \lambda(e)\mathrm{per}(A)$ for each $A \in \mathcal{H}_n$.*

As noted previously, this conjecture is now known to hold for $n \le 9$.

3. THE IRREDUCIBLE CHARACTERS OF S_n

There is a natural bijective correspondence between the partitions of n and the irreducible characters of S_n that is given by a formula. This formula, which we present in this section, has been particularly helpful during attempts to obtain inequalities among immanants. We begin with the necessary definitions and notation.

A partition of n is a nonincreasing sequence, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$, of positive integers such that $\sum_{j=1}^{s} \alpha_j = n$. We let $\mathcal{P}_n$ denote the set of all partitions of n. Associated with each $\alpha \in \mathcal{P}_n$ is a node diagram, sometimes called a Young diagram, which is basically an array of dots such that the number of dots on the ith row is precisely the ith term of α. For example, if $\alpha = (4^2, 3, 2^3, 1) = (4, 4, 3, 2, 2, 2, 1)$, then the associated node diagram is

$$\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & & & \end{array}$$

Also associated with partitions are objects known as α-tableaux. If $\alpha \in \mathcal{P}_n$, one obtains an α-tableaux from α's node diagram by injectively replacing the dots in α's node diagram with the integers 1, 2,$\ldots, n$. For example, if $\alpha = (4^2, 3, 2^3, 1)$, then an α-tableau is

$$\begin{array}{cccc} 2 & 4 & 6 & 9 \\ 11 & 13 & 12 & 1 \\ 5 & 3 & 7 & \\ 10 & 16 & & \\ 17 & 15 & & \\ 18 & 8 & & \\ 14 & & & \end{array}$$

Clearly there are $n!$ α-tableaux.

Suppose $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ is a partition of n, and T is an associated α-tableau. Let $\Delta_1, \Delta_2, \ldots, \Delta_s$ be the row sets of T. For example, if T is the tableau displayed above, then the row sets are the sets $\{2, 4, 6, 9\}$, $\{11, 13, 12, 1\}$, $\{5, 3, 7\}$, $\{10, 16\}$, $\{17, 15\}$, $\{18, 8\}$, and $\{14\}$. The row group $\mathcal{R}_T$, associated with α is the set of all $\sigma \in \mathrm{S}_n$ such that $\sigma(\Delta_i) = \Delta_i$ for each i, $1 \leq i \leq s$. The column group $\mathcal{C}_T$ is defined analogously with respect to the column sets of T. The row symmetrizer r_T is then $\sum_{\sigma \in \mathcal{R}_T} \sigma$, and the column antisymmetrizer is $\sum_{\tau \in \mathcal{C}_T} \epsilon(\sigma)\sigma$. The product r_T and c_T (in either order) is a near-idempotent known as a Young symmetrizer.

If $\alpha \in \mathcal{P}_n$, then we let λ_α denote the irreducible character of S_n associated with α. The above-mentioned formula for λ_α, which appears on

p. 108 of [11], is then

$$\lambda_\alpha = \frac{\lambda_\alpha(e)}{n!} \sum_{\sigma \in S_n} \sigma^{-1} r_T c_T \sigma. \tag{3.1}$$

The number $\lambda_\alpha(e)$, a positive integer, is the degree of λ_α, and may be computed using the well-known hook formula, which may be found in [6].

If $f, g \in \mathbb{C}S_n$, then, by Lemma 2 of [15], we have

$$\sum_{\sigma \in S_n} \sigma^{-1} f g \sigma = \sum_{\sigma \in S_n} \sigma^{-1} g f \sigma. \tag{3.2}$$

Applying (3.2) to (3.1), we obtain the following alternative expressions for λ_α:

$$\lambda_\alpha = \frac{\lambda_\alpha(e)}{n!} \sum_{\tau \in S_n} \tau^{-1} c_T r_T \tau, \tag{3.3}$$

$$\lambda_\alpha = \frac{\lambda_\alpha(e)}{n!|\mathcal{R}_T|} \sum_{\tau \in S_n} \tau^{-1} r_T c_T r_T \tau, \tag{3.4}$$

$$\lambda_\alpha = \frac{\lambda_\alpha(e)}{n!|\mathcal{C}_T|} \sum_{\tau \in S_n} \tau^{-1} c_T r_T c_T \tau. \tag{3.5}$$

Equations (3.4) and (3.5) appear as (2.6) and (2.7) in [15] and have been particularly useful in obtaining inequalities between pairs of normalized immanants.

Suppose T is a tableau associated with partition α of n. Let ζ_T denote $r_T c_T r_T$, and let η_T denote $c_T r_T c_T$. Observe that r_T and c_T are Hermitian, and that $r_T^2 = |\mathcal{R}_T| r_T$ and $c_T^2 = |\mathcal{C}_T| c_T$; hence

$$\zeta_T = r_T c_T r_T = |\mathcal{C}_T|^{-1} r_T c_T c_T r_T = |\mathcal{C}_T|^{-1} (c_T r_T)^* (c_T r_T). \tag{3.6}$$

Consequently, ζ_T is positive semidefinite Hermitian. Similarly, one can show that

$$\eta_T = |\mathcal{R}_T|^{-1} (r_T c_T)^* (r_T c_T); \tag{3.7}$$

hence η_T is also positive semidefinite Hermitian. This means that if we let ξ_T denote the Young symmetrizer $c_T r_T$, then we have

$$\lambda_\alpha = \frac{\lambda_\alpha(e)}{n!|\mathcal{R}_{T'}||\mathcal{C}_{T'}|} \sum_T \xi_T^* \xi_T = \frac{\lambda_\alpha(e)}{n!|\mathcal{R}_{T'}|} \sum_T \zeta_T, \tag{3.8}$$

where the summations are over all α-tableaux T, and T' is an arbitrary α-tableau. The objects ζ_T and η_T might well be called alternative Young symmetrizers, since these could be substituted for the standard Young symmetrizers ξ_T in any development of the representation theory of the symmetric groups.

Now, suppose β is some other partition of n, and let T' denote a β-tableau. One might now ask whether inequalities in the form $\widehat{\zeta_T} \leq \widehat{\zeta_{T'}}$ or $\widehat{\zeta_T} \leq \widehat{\eta_{T'}}$ hold. Of course one could not expect such inequalities to hold for an arbitrary β-tableau T'. However, it may be possible to find a bijection φ between the α-tableau and the β-tableaux such that $\widehat{\zeta_T} \leq \widehat{\zeta_{\varphi(T)}}$ for each α-tableaux T, and indeed the proof of Theorem 8 below could be viewed in this way. Moreover, the proof of Theorem 7 is based on inequalities among the η_T and various related objects. Thus the immanant inequalities presented in Theorems 7 and 8 are actually obtained by summing inequalities involving the alternative Young symmetrizers. These other inequalities, though stated in [15], [17], and [18], have otherwise received very little attention.

Critical to the proofs of the results of the next section is a special type of function (in $\mathbb{C}S_n$) which is nonnegative on the members of $\mathcal{H}_n$. The ψ-functions of Heyfron (see [2], [3], or [20]) are functions of this type. Similar objects are used by the author in [14], [15], [17], and [18]. The basic idea goes back to Neuberger [12], though a somewhat weaker form may be inferred from [8]. For the bottom line on this idea see Lemma 5 of [15].

4. SOME RECENT RESULTS

If $\alpha, \beta \in \mathcal{P}_n$, then we write $\alpha \leq \beta$ if $\widehat{\lambda}_\alpha \leq \widehat{\lambda}_\beta$. Equivalently, we write $\alpha \leq \beta$ if $[\lambda_\alpha(e)]^{-1} d_{\lambda_\alpha}(A) \leq [\lambda_\beta(e)]^{-1} d_{\lambda_\beta}(A)$ for each $A \in \mathcal{H}_n$.

LEMMA 2. *The relation $\leq$ is a partial order on $\mathcal{P}_n$.*

At first glance this may seem obvious. However, to prove the above it is necessary to show that if α and β are partitions of n and $[\lambda_\alpha(e)]^{-1} d_{\lambda_\alpha}(A)$ $= [\lambda_\beta(e)]^{-1} d_{\lambda_\beta}(A)$ for each $A \in \mathcal{H}_n$, then $\alpha = \beta$. For a proof see [16].

Since the immanant associated with the partition (1^n) is the determinant function det( ), Theorem 1 guarantees us that if $\alpha \in \mathcal{P}_n$ then $\alpha \geq (1^n)$. The immanant associated with (n), as is easily seen from (3.1), is the permanent function. Hence, Conjecture 3, the permanental dominance conjecture for immanants, is that $\alpha \leq (n)$ for each $\alpha \in \mathcal{P}_n$. It is the goal of this author to find a characterization of $\leq$ that is reasonably simple. The ideal would be a characterization that depends upon finitely many conditions that are reasonably easy to check. We therefore are not restricting our attention to the extreme partitions, (n) and (1^n), but wish to be able to resolve the question of inequality between any pair of normalized immanants. Of course the solution of this broader problem would resolve Conjecture 3 immediately.

Merris and Watkins [10] were perhaps the first to attempt to discover inequalities between pairs of nonextreme normalized immanants. After showing that $(n) > (n-1,1)$ and $(3,1^{n-3}) > (2,1^{n-2}) > (1^n)$, they conjectured that the hook partitions, partitions of the form $(n-k,1^k)$ where $0 \leq k \leq n$, decrease monotically. In other words, Merris and Watkins conjectured that $(n-k,1^k) \geq (n-k-1,1^{k+1})$ whenever $0 \leq k \leq n-1$. This conjecture was considered by virtually everyone in the area before being resolved by P. Heyfron [2], who proved the following.

THEOREM 3 (P. Heyfron). *If n is a positive integer and k is a nonnegative integer not exceeding $n-1$, then $(n-k,1^k) \geq (n-k-1,1^{k+1})$.*

After the publication of the Merris and Watkins's [10] but before the publication of Theorem 3, James and Liebeck [7] proved the following.

THEOREM 4. *If $\alpha = (p,q)$ is a partition of n, then $\alpha \leq (n)$. Equivalently, if α is a partition with only two terms, then $[\lambda_\alpha(e)]^{-1} d_{\lambda_\alpha}(A) \leq \mathrm{per}(A)$ for each $A \in \mathcal{H}_n$.*

This theorem provided evidence for Conjecture 3, the permanental dominance conjecture for immanants: at least it holds for partitions having two or fewer terms. Actually, Theorem 4 follows from a more general result, as shown by the author in [19].

The following theorem not only generalized both Heyfron's Theorem 3 and Schur's theorem applied to immanants, but provided a tremendous amount of new information about the relationship between pairs of normalized immanants. See [15] and [16].

THEOREM 5 (T. Pate). *Suppose $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_t)$ is a partition of n, and s is a positive integer not exceeding t such that if $s < t$ then $\alpha_s > \alpha_{s+1}$, and $\alpha_i = \alpha_s$ for each $1 \leq i \leq s$. Let β denote the partition $(\alpha_1 - 1, \alpha_2 - 1, \ldots, \alpha_s - 1, \alpha_{s+1}, \ldots, \alpha_s, 1^t)$. Then $\alpha \geq \beta$. Equivalently, $[\lambda_\alpha(e)]^{-1} d_{\lambda_\alpha}(A) \geq [\lambda_\beta(e)]^{-1}\, d_{\lambda_\beta}(A)$ for each $A \in \mathcal{H}_n$.*

During 1989 the following generalization of Theorem 5 was obtained. See [16].

THEOREM 6 (T. Pate). *Suppose $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_t)$ is a partition of n, and s is a positive integer not exceeding t such that if $s < t$ then $\alpha_s > \alpha_{s+1}$. Let β denote the partition $(\alpha_1 - 1, \alpha_2 - 1, \ldots, \alpha_s - 1, \alpha_{s+1}, \ldots, \alpha_s, 1^t)$. Then $\alpha \geq \beta$. Equivalently, $[\lambda_\alpha(e)]^{-1} d_{\lambda_\alpha}(A) \geq [\lambda_\beta(e)]^{-1} d_{\lambda_\beta}(A)$ for each $A \in \mathcal{H}_n$.*

Note that Theorem 6 is simply Theorem 5 with the condition $\alpha_1 = \alpha_2 = \cdots = \alpha_s$ deleted. With respect to node diagrams this means that in passing from the α-diagram to the β-diagram one moves any column of the α-diagram (except the first) to the end of the first column.

The following theorem, which first appeared in [17], implies both Theorems 5 and 6. Instead of moving whole columns of nodes, we see that we may, under certain conditions, move nodes one at a time.

THEOREM 7 (T. Pate). *Suppose $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_t)$ is in $\mathcal{P}_n$ and $1 \leq s \leq t$. If $s < t$, then suppose $\alpha_s > \alpha_{s+1}$. Let β denote the partition $(\alpha_1, \alpha_2, \ldots, \alpha_s - 1, \alpha_{s+1}, \ldots, \alpha_t, 1)$. Then $\alpha \geq \beta$. Equivalently, $[\lambda_\alpha(e)]^{-1} d_{\lambda_\alpha}(A) \geq [\lambda_\beta(e)]^{-1} d_{\lambda_\beta}(A)$ for each $A \in \mathcal{H}_n$.*

With respect to node diagrams, Theorem 7 implies that in passing from α to β we simply delete one of the corner nodes of the α-diagram and insert a new node at the end of the first column. Obviously, Theorem 7 implies both Theorem 3 and Theorem 5. But Theorem 7 also implies Theorem 6. For a discussion see [17].

EXAMPLE 1. If $\alpha = (5, 4^2, 3, 2)$, then Theorem 7 implies that $\alpha \geq \beta$, where β is $(4^3, 3, 2, 1), (5, 4, 3^2, 2, 1), (5, 4^2, 2^2, 1)$, or $(5, 4^2, 3, 1^2)$. The α-diagram with corner nodes enlarged is

$$\begin{array}{ccccc}
\cdot & \cdot & \cdot & \cdot & \bullet \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \bullet & \\
\cdot & \cdot & \bullet & & \\
\cdot & \bullet & & &
\end{array}$$

The node diagrams for the above-listed admissible β's are obtained by moving one of the enlarged nodes to the end of the first column of the diagram.

EXAMPLE 2. If we start with $\alpha = (2^6)$ and apply Theorem 7 repeatedly, we can obtain

$$\alpha > (2^5, 1^2) > (2^4, 1^4) > (2^3, 1^6) > (2, 1^{10}) > (1^{12}).$$

If we apply Theorem 7 repeatedly starting with $\alpha = (3^3)$, then we can obtain

$$(3^3) > (3^2, 2, 1) > (3, 2^2, 1^2) > (2^3, 1^3).$$

Moreover, other sequences are possible, for Theorem 7 also implies that

$$(3^3) > (3^2, 2, 1) > (3^2, 1^3) > (3, 2, 1^4).$$

For some time Conjecture 3 was unresolved in case $n = 8$ only because it was not known whether $(2^4) \leq (8)$. Indeed, Theorems 7 and 4 imply that $\alpha \leq (8)$ for any $\alpha \in \mathcal{P}_8$ other than (2^4). The following theorem, which appears in [18], implies that $(2^4) \leq (8)$, thus completing the verification of the permanental dominance conjecture for immanants in case $n = 8$.

THEOREM 8 (T. Pate). *Suppose* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s, 1^t)$ *is a partition of* n *such that* $s > 1$ *and* $\alpha_s = 2$. *Let* β *denote the partition* $(\alpha_1 + 2, \alpha_2, \ldots, \alpha_{s-1}, 1^t)$. *Then* $\alpha \leq \beta$. *Equivalently,* $[\lambda_\alpha(e)]^{-1} d_{\lambda_\alpha}(A) \leq [\lambda_\beta(e)]^{-1} d_{\lambda_\beta}(A)$ *for each* $A \in \mathcal{H}_n$.

EXAMPLE 3. If we apply Theorem 8 repeatedly starting with $\alpha = (2^4)$, we obtain the sequence

$$(2^4) < (4, 2^2) < (6, 2) < (8).$$

Theorem 8, in conjunction with Theorem 7 and Theorem 4, implies the following.

THEOREM 9 (T. Pate). *If* α *is a partition of* n *of the form* $(p, q, 2^r, 1^s)$, *where* p, q, r, *and* s *are nonnegative integers, then* $\alpha \leq (n)$.

Since all partitions of n where $n \leq 8$ are of the form $(p, q, 2^r, 1^s)$, the permanental dominance conjecture for immanants must hold whenever $n \leq 8$. If $n = 9$, then there is only one partition that is not of the form $(p, q, 2^r, 1^s)$, namely (3^3). But P. Heyfron has communicated privately to the author that he and G. D. James have worked out this case separately. Hence, Conjecture 3 may be considered to be true whenever $n \leq 9$.

EXAMPLE 4. Suppose $\alpha = (6, 4, 2^3, 1^3) \in \mathcal{P}_{19}$. Then Theorem 7 implies

$$\alpha < (7, 4, 2^3, 1^2) < (8, 4, 2^3, 1) < (9, 4, 2^3).$$

Theorem 8 now implies that

$$(9,4,2^3) < (11,4,2^2) < (13,4,2) < (15,4),$$

and Theorem 4 implies that $(15,4) < (19)$. Hence, $\alpha < (19)$.

Recently, in an effort to extend Theorem 9 to partitions of the form $(p,q,3^r,2^s,1^t)$, Heyfron [5] has obtained the following theorems. Heyfron's proofs depend upon Theorem 8, various properties of the ψ-functions, and an explicit formula for the value of an irreducible character at a transposition.

THEOREM 10 (P. Heyfron). *If p and q are positive integers with $p \geq 2q+2$, then $(p+3, p^{q-1}) \geq (p^q, 3)$.*

THEOREM 11 (P. Heyfron). *Suppose $\alpha = (x, 3^p, 2^q, 1^z)$ is a partition of n, where $x \geq 3$ and p, q, and z are nonnegative integers such that $n \geq 5p+2$. Then $\alpha \leq (n)$.*

THEOREM 12 (P. Heyfron). *Suppose that $\alpha = (x, y, 3, 2^q, 1^z)$ is a partition of n, where x, y, z, and q are nonnegative integers such that $y+z \geq x \geq 6$. Then $\alpha \leq (n)$.*

Unfortunately, Theorem 11 requires that n be large relative to p; hence, in a relative sense, the partition α cannot have very many parts of size 3. For example, the partition (4, 3, 3) is the only partition of 10 that is not of the form $(p,q,2^r,1^s)$, but in this case $p = 2$ and $5p+2 = 12$, so Theorem 11 fails to apply. The same problem arises in case $\alpha = (3^3)$. Theorem 10 also yields no information about (3^3), because in this case $p = 3$ and $q = 2$ but $5p+2 = 17$, so $n \geq 17$ fails to be true. Moreover, Theorem 12 requires that the first term of α be 6 or more. Hence, this Theorem also provides no information about (3^3) or $(4,3^2)$. However, if $\alpha = (6^2,3)$, then Theorem 10 implies that $\alpha < (9,6)$ and Theorem 4 gives $(9,6) \leq (15)$. Hence, $(6^2,3) \leq (15)$. Note that Theorem 12 implies $(6^2,3) \leq (15)$ directly.

The limitations of Theorems 10, 11, and 12 are probably due to the fact that Theorem 8 plays an essential role in their proofs. Recall that Theorem 8 involves moving parts of size 2 only.

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